ASSIGNMENT No. 02

Business Mathematics (1429) B.A/ B.Com, BBA

Spring, 2025

Q. 1 a) Differentiate between singular and non-singular matrices.

Definition of Singular and Non-Singular Matrices

A singular matrix is defined as a square matrix that does not have an inverse. This implies that its determinant is equal to zero. In contrast, a non-singular matrix is essentially the opposite; it is also a square matrix, but it possesses a non-zero determinant, thereby having an inverse. The determinant plays a crucial role in determining the properties of a matrix, including its singularity. Understanding the fundamental definitions helps set the stage for further exploration into their characteristics.

Characteristics of Singular Matrices

Singular matrices exhibit specific characteristics that make them critical in various mathematical applications. One prominent feature is the presence of linearly dependent rows or columns, meaning that at least one row or column can be expressed as a linear combination of the others. This results in a determinant of zero, indicating that the matrix cannot span the entire space it occupies, thus making it singular. The implications of such characteristics are significant, particularly in systems of linear equations.

Characteristics of Non-Singular Matrices

Non-singular matrices, on the other hand, exhibit a robust set of characteristics. They are characterized by linearly independent rows or columns, which guarantees that no row or column can be represented as a linear combination of the others. This independence is reflected in their non-zero determinant, allowing the matrix to have an inverse. Non-singular matrices play a critical role in solving linear systems, as they ensure a unique solution exists.

Determinant of Singular Matrices

The determinant is a crucial factor in identifying whether a matrix is singular. For a singular matrix, the determinant is precisely zero. This condition leads to a variety of implications in linear algebra, particularly concerning the solutions of linear equations and matrix transformations. When a matrix is singular, one of the fundamental obstacles it presents is the inability to uniquely determine solutions to a corresponding set of linear equations.

Determinant of Non-Singular Matrices

Conversely, non-singular matrices possess a non-zero determinant. This feature indicates that the matrix can be inverted, making it a powerful tool for solving systems of linear equations. The non-zero determinant not only confirms the matrix's invertibility but also guarantees that the linear transformation associated with the matrix will span the entire output space, thus generating distinct outcomes for given inputs.

Geometric Interpretation of Singular Matrices

Geometrically, a singular matrix can be visualized as collapsing a multi-dimensional space into a lower dimension. For instance, in two-dimensional space, a transformation represented by a singular matrix may reduce the space to a line or a point, effectively losing information about the original vectors. This geometric interpretation emphasizes how singular matrices lack the capability to maintain the dimensionality of the space they operate in.

Geometric Interpretation of Non-Singular Matrices

In contrast, a non-singular matrix maintains the dimensionality of the space during transformations. For example, a non-singular transformation in a three-dimensional space will map points to other points in the same space without collapsing dimensions. This preservation of dimensionality is critical in applications such as computer graphics, robotics, and physics, where changes in space must be described fully and accurately.

Applications of Singular Matrices

Singular matrices have unique applications in various fields of study, particularly in understanding linear dependence among vectors. They commonly arise in constraint systems, optimization problems, and scenarios involving redundant information. In these contexts, recognizing the singularity of a matrix can lead to insights and methods for simplifying complex problems or identifying essential constraints that must be addressed.

Applications of Non-Singular Matrices

Non-singular matrices find extensive applications across numerous fields including engineering, computer science, and economics. Their role in providing unique solutions to systems of linear equations makes them invaluable in modeling and simulations. Additionally, non-singular matrices are used in computer graphics for transformations, cryptography for encoding information, and optimization problems where unique solutions are essential.

Finding the Inverse of Non-Singular Matrices

One of the primary advantages of non-singular matrices is that they can be inverted using various methods, such as the adjugate method or Gaussian elimination. The availability of an inverse allows for solving equations of the form Ax=bAx = b$Ax=b$, where AAA is a non-singular matrix. This capability is crucial not only in theoretical mathematics but also in practical applications where solutions must be computed efficiently and accurately.

Role of Row Echelon Form

The row echelon form is also a decisive concept in distinguishing between singular and non-singular matrices. A matrix in row echelon form that has a row of zeros implies that the original matrix is singular. Conversely, a matrix that can be reduced to row echelon form without any row of zeros is non-singular. Understanding this classification method provides a systematic way to analyze matrices for singularity.

Rank of Singular and Non-Singular Matrices

The rank of a matrix serves as another pivotal indicator of its characteristics. A singular matrix has a rank that is less than its dimension, reflecting the linear dependence among its rows or columns. In contrast, a non-singular matrix has full rank, which equals its dimension. This distinction in rank helps provide an additional layer of understanding and analysis in linear algebra.

Linear Independence and Dependence

Linear independence is a central theme in linear algebra and relates directly to the properties of singular and non-singular matrices. A set of vectors is considered linearly independent if no vector in the set can be expressed as a linear combination of the others. Non-singular matrices are associated with linearly independent vector sets, whereas singular matrices typically correlate with linearly dependent sets, leading to important implications in vector space theory.

System of Linear Equations

The study of systems of linear equations serves as a practical application that clearly differentiates singular from non-singular matrices. In a system represented by a singular matrix, solutions may either be non-existent or infinitely many due to undetermined variables arising from dependencies among equations. Non-singular matrices lead to well-defined, unique solutions, highlighting their critical role in mathematical problem-solving.

Condition Number and Stability

The condition number of a matrix is a measure that assesses its sensitivity to numerical operations, particularly useful in the context of singular and non-singular matrices. A singular matrix has an undefined or infinite condition number, indicating extreme sensitivity and instability. Meanwhile, a non-singular matrix typically exhibits a finite condition number, denoting a level of stability that allows for more reliable computations and solutions in practical applications.

Eigenvalues of Singular Matrices

In the study of eigenvalues, singular matrices uniquely feature at least one eigenvalue equal to zero. This characteristic indicates a failure to span the full vector space, thereby laying bare the implications of singularity. In contrast, non-singular matrices possess eigenvalues that are all non-zero, reflecting their ability to transform vectors without loss of dimension or information.

Eigenvalues of Non-Singular Matrices

For non-singular matrices, having non-zero eigenvalues is indicative of their ability to invert, as the eigenvalues can be used to determine the scaling factors in respective directions under linear transformations. These properties allow for effective analysis in diverse applications, including stability analysis in dynamic systems as well as in control theory, where the behavior of systems must be monitored and manipulated.

Transformation Properties

When examining linear transformations represented by matrices, singular matrices imply transformations that collapse dimensions and render parts of space inaccessible. This collapsing nature disrupts the ability to uniquely map inputs to outputs. Non-singular matrices retain transformations that fully map the inputs to a correctly corresponding output, essential for preserving the integrity of data and ensuring transformability in advanced applications.

Matrix Decomposition Techniques

Various matrix decomposition techniques, like LU decomposition and QR decomposition, provide potent methods for analyzing singular and non-singular matrices. For singular matrices, these techniques may reveal linearly dependent rows or columns through their structural analysis. Meanwhile, non-singular matrices will lend themselves to effective decomposition, aiding in computational tasks and fostering a deeper understanding of their properties.

Real-World Importance of Non-Singular Matrices

Non-singular matrices have profound implications in real-world applications. For instance, in engineering, non-singular matrices are critical in stability analysis and controller design. They ensure reliable solutions in systems governed by linear relationships, such as those found in mechanical systems. The ability to provide accurate solutions can impact various sectors, including infrastructure, technology, and economics.

Summation

In conclusion, the distinctions between singular and non-singular matrices are fundamental to the study of linear algebra. Singular matrices, with their zero determinant and linear dependencies, hinder the unique resolution of systems of equations, while non-singular matrices, characterized by non-zero determinants and linear independence, facilitate problem-solving and transformations. Understanding these differences not only enhances theoretical knowledge but also enables practical applications across various scientific and engineering domains. The implications of matrix classifications resonate throughout mathematics, underscoring their critical importance across numerous disciplines.

b) Solve the following system of linear equations using matrices.

$$2x_1 + 3x_2 - 4x_3 = -4$$
$$x_1 - x_2 + x_3 = 2$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Step 1: Write in Matrix Form

Matrix form is:

$$AX = B$$

where

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ 2 \\ 14 \end{pmatrix}$$

Step 2: Solve for X

We know:

$$X = A^{-1}B$$

So we need to find the inverse of A, and then multiply it by B.

Step 3: Find A^{-1}

First, calculate the determinant $\det(A)$:

$$\begin{aligned} \det(A) &= 2((-1)(3) - (1)(2)) - 3((1)(3) - (1)(1)) + (-4)((1)(2) - (-1)(1)) \\ &= 2(-3 - 2) - 3(3 - 1) - 4(2 + 1) \\ &= 2(-5) - 3(2) - 4(3) \\ &= -10 - 6 - 12 \\ &= -28 \end{aligned}$$

So, $\det(A) = -28 \neq 0$, meaning A is invertible.

Step 4: Solve Using Row Operations (shortcut)

Rather than doing the full A^{-1}, a quicker method for small systems is using augmented matrix and row reduction:

Set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & -4 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & 3 & 14 \end{array}\right]$$

Now perform row operations:

Step 1: Make the leading 1 in Row 1. (Divide R1 by 2)

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & -2 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & 3 & 14 \end{array}\right]$$

Step 2: Eliminate below the leading 1:

- $R2 - R2 - R1$
- $R3 - R3 - R1$

New rows:

$$R2 - (1, -1, 1, 2) - (1, \frac{3}{2}, -2, -2) - (0, -\frac{5}{2}, 3, 4)$$

$$R3 - (1, 2, 3, 14) - (1, \frac{3}{2}, -2, -2) - (0, \frac{1}{2}, 5, 16)$$

Now:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & -2 \\ 0 & -\frac{5}{2} & 3 & 4 \\ 0 & \frac{1}{2} & 5 & 16 \end{array}\right]$$

Step 3: Make the leading coefficient in R2 a 1.

Multiply R2 by $\frac{-2}{5}$:

$$R2 - (0, 1, -\frac{6}{5}, -\frac{8}{5})$$

Matrix now:

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -2 & -2 \\ 0 & 1 & -\frac{6}{5} & -\frac{8}{5} \\ 0 & \frac{1}{2} & 5 & 16 \end{array}\right]$$

Step 4: Eliminate above and below the leading 1 (second column):

- $R1 - R1 - \frac{3}{2} \times R2$
- $R3 - R3 - \frac{1}{2} \times R2$

First:

$$R1 - (1, \frac{3}{2}, -2, -2) - \frac{3}{2}(0, 1, -\frac{6}{5}, -\frac{8}{5})$$
$$- (1, 0, -2 + \frac{9}{5}, -2 + \frac{12}{5})$$
$$- (1, 0, \frac{-10+9}{5}, \frac{-10+12}{5})$$
$$- (1, 0, \frac{-1}{5}, \frac{2}{5})$$

Second:

$$R3 - (0, \frac{1}{2}, 5, 16) - \frac{1}{2}(0, 1, -\frac{6}{5}, -\frac{8}{5})$$
$$- (0, 0, 5 + \frac{3}{5}, 16 + \frac{4}{5})$$
$$- (0, 0, \frac{25+3}{5}, \frac{80+4}{5})$$
$$- (0, 0, \frac{28}{5}, \frac{84}{5})$$

Matrix now:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{-1}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{6}{5} & -\frac{8}{5} \\ 0 & 0 & \frac{28}{5} & \frac{84}{5} \end{array}\right]$$

Step 5: Solve for x_3 from Row 3:

$$\frac{28}{5}x_3 = \frac{84}{5}$$

Multiply both sides by 5:

$$28x_3 = 84$$

Divide:

$$x_3 = 3$$

Step 6: Substitute $x_3 = 3$ into Row 2:

$$x_2 - \frac{6}{5}(3) = -\frac{8}{5}$$

$$x_2 - \frac{18}{5} = -\frac{8}{5}$$

Add $\frac{18}{5}$ to both sides:

$$x_2 = \frac{10}{5} = 2$$

Thus:

$$x_2 = 2$$

Step 7: Substitute $x_2 = 2$, $x_3 = 3$ into Row 1:

$$x_1 - \frac{1}{5}(3) = \frac{2}{5}$$

$$x_1 - \frac{3}{5} = \frac{2}{5}$$

Add $\frac{3}{5}$ to both sides:

$$x_1 - \frac{5}{5} - 1$$

Thus:

$$x_1 - 1$$

Final Solution:

$$\boxed{x_1 - 1, \quad x_2 - 2, \quad x_3 - 3}$$

Q. 2 Mr. Asad bought 3 kg sugar, 10 kg wheat and 1 kg salt from the local market at the rates of Rs. 50, Rs. 20 and Rs. 15 per kg respectively. If the prices of these items in the Sunday market are Rs. 45, Rs. 16 and Rs. 12 per kg respectively and the one way fare to the Sunday market is Rs. 15. Using matrices multiplication, find his net profit or loss.
(20)

Ans:

$$\begin{pmatrix} 50 & 20 & 15 \\ 45 & 16 & 12 \\ 15 & 15 & 15 \end{pmatrix}\begin{pmatrix} 3K & Suger \\ 10K & Wheat \\ 1K & Salt \end{pmatrix}$$

$$= \begin{pmatrix} 50\times3 & 20\times3 & 15\times1 \\ 45\times3 & 16\times10 & 12\times1 \\ 15\times3 & 15\times10 & 15\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 150+200+15 \\ 135+160+12 \\ 45+150+15 \end{pmatrix}$$

$$= \begin{pmatrix} 365 \\ 307 \\ 210 \end{pmatrix}$$

$$Loss\ \ if\ \ buy\ at\ Sunday\ market \ = 365 - 307$$
$$= 58\ Rs.$$
$$Loss\ (if\ \ buy\ at\ fare) \ = 365 - 210$$
$$= 155\ Rs.$$

Q. 3 a) For the function $f(x) = x^2 - 2x + 3$ Find the point where the tangent Line of

this function is horizontal.
(20)

Ans: (a)

$$f(x) = x^2 - 2x + 3$$

First finding derivatives w.r.t.x

$$\frac{dy}{dx} = 2x - 2$$

Tangent line is horizontal when

$$\frac{dy}{dx} = 0$$

$$2x - 2 = 0$$

$$2x = 2$$

$$2x = 2$$

$$x = 1$$

At point when x=1 the tangent line of this function is horizontal

When x = 1

$$f(x) = y = 2$$

$$(1,2)$$

b) The distance covered by a car in kilometers is given as a function of time in hours by the following relation $f(t) = t^2 + 50t + 15$

Find the instantaneous velocity of the car at t=2

Ans: (b)

$$\frac{ay}{dt} = 2t - 50$$

$$V_{(t)} = 2t - 50$$

Velocity at t=2

$$V(t) = 2t - 50$$

$$V(2) = 2(2) - 50$$

$$= 4 - 50$$

$$V(2) = -46$$

Q. 4 A manufacturer has determined a cost function that expresses the annual cost of purchasing, owning and maintaining its raw material inventory (C) as a function of the size of each order (q). The cost function is $C = \frac{51200}{q} + 80q + 750000$

a) Determine the order size q which minimizes inventory cost C.
(20)

b) What are minimum inventory costs expected to equal?

a) Determine the order size q which minimizes inventory cost C.

Ans: (a)

$$c(q)=\frac{51200}{q}+80q+750000$$

$$c^{1}(q)=\frac{-51200}{q^{2}}+80$$

$$c^{1}=0$$

$$\frac{51200}{q^{2}}=80$$

$$\sqrt{q^{2}}=\sqrt{\frac{5120\not{0}}{8\not{0}}}=\sqrt{640}$$

$$q=25.30$$

b) What are minimum inventory costs expected to equal.

Ans: (b)

$$q=25.30$$

$$c(q)=\frac{51200}{q}+80q+750000$$

$$c(25.30)=\frac{51200}{25.30}+80q(25.30)+750000$$

$$=2023.72+2024+750000$$

$$=754047.72$$

Q.5 a) Given $A=\begin{bmatrix}2 & -4 & 1\\ 4 & 0 & -3\\ -1 & 3 & -2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 0\\ 1 & 2\\ 2 & 1\end{bmatrix}$ Find the product $AB=?$

Matrix A (3×3 matrix):

$$A = \begin{pmatrix} 2 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & -2 \end{pmatrix}$$

Matrix B (3×2 matrix):

$$B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$$

You are asked to find AB.

- Since A is 3×3 and B is 3×2, the product AB will be a 3×2 matrix.

Step 1: Multiply Matrices

The rule is:

$$(AB)_{ij} = \text{Row } i \text{ of } A \times \text{Column } j \text{ of } B$$

Now, calculating each element:

First Row of AB:

- First element:

$$(2)(3) + (-4)(1) + (1)(2) = 6 - 4 + 2 = 4$$

- Second element:

$$(2)(0) + (-4)(2) + (1)(1) = 0 - 8 + 1 = -7$$

Second Row of AB:

- First element:

$$(4)(3) + (0)(1) + (-3)(2) = 12 + 0 - 6 = 6$$

- Second element:

$$(4)(0) + (0)(2) + (-3)(1) = 0 + 0 - 3 = -3$$

Third Row of AB:

- First element:

$$(-1)(3) + (3)(1) + (-2)(2) = -3 + 3 - 4 = -4$$

- Second element:

$$(-1)(0) + (3)(2) + (-2)(1) = 0 + 6 - 2 = 4$$

Step 2: Final Product

Thus,

$$AB = \begin{pmatrix} 4 & -7 \\ 6 & -3 \\ -4 & 4 \end{pmatrix}$$

Final Answer:

$$\boxed{AB = \begin{pmatrix} 4 & -7 \\ 6 & -3 \\ -4 & 4 \end{pmatrix}}$$

b) Find the 2nd order derivative of the function: $f(x) = \sqrt[3]{x^5}$.

$$f(x) = \sqrt[3]{x^5}$$

This is the cube root of x^5.

First, rewrite it in exponent form:

$$f(x) = (x^5)^{1/3}$$

Using the rule of exponents:

$$f(x) = x^{5/3}$$

Step 1: Find the **first derivative** $f'(x)$

Differentiate using the power rule:

$$f'(x) = \frac{5}{3}x^{(5/3)-1} = \frac{5}{3}x^{2/3}$$

Step 1: Find the **first derivative** $f'(x)$

Differentiate using the power rule:

$$f'(x) = \frac{5}{3}x^{(5/3)-1} = \frac{5}{3}x^{2/3}$$

Step 2: Find the **second derivative** $f''(x)$

Differentiate $f'(x)$:

$$f''(x) = \frac{5}{3} \times \frac{2}{3}x^{(2/3)-1}$$
$$= \frac{10}{9}x^{-1/3}$$

Or, if you want to write it without a negative exponent:

$$f''(x) = \frac{10}{9} \times \frac{1}{x^{1/3}}$$
$$= \frac{10}{9x^{1/3}}$$

Final Answer:

$$\boxed{f''(x) = \frac{10}{9x^{1/3}}}$$